Exploring a Parametric Curve

a) Describe the curve traced out by the parametrization:

$$\begin{aligned} x &= t\cos t \\ y &= t\sin t, \end{aligned}$$

where $0 \leq t \leq 4\pi$.

b) Set up and simplify, but do not integrate, an expression for the arc length $\displaystyle\int_0^{4\pi} \frac{ds}{dt}\, dt$ of this curve.

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a) $x^2+y^2 = t^2\cos^2 t + t^2\sin^2 t$

$= t^2$

$t=0 \Rightarrow x=0,\ y=0$

$\therefore$ It is a spiral with increasing radius t starting at the origin, with 2 revolutions from 0 to 4π around the origin.

b) $ds = \sqrt{dx^2+dy^2}$

$\Rightarrow \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$

$\frac{dy}{dt} = \sin t + t\cos t$

$\frac{dx}{dt} = \cos t + t(-\sin t)$

$= \cos t - t\sin t$

$$\int_0^{4\pi} \frac{ds}{dt}\,dt = \int_0^{4\pi} \sqrt{\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t}\; dt$$

$$= \int_0^{4\pi} \sqrt{1+t^2(\sin^2 t+\cos^2 t)}\; dt$$

$$= \int_0^{4\pi} \sqrt{1+t^2}\; dt$$